Chapter 8: *More Applications.*

8.1 Arc Length (we already discussed)

8.3 Center of Mass

For your own interest read (not on test):

8.2: Surface area

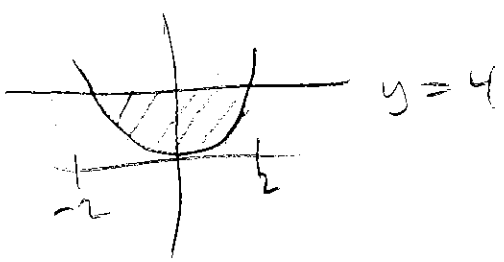
8.3: Hydrostatic (water) pressure & force

8.4: Economics and biology apps

8.5: Probability apps (bell curve)

8.3 Center of Mass

Goal: Given a thin plate (a *lamina*) where the mass is uniformly distributed, we find the center of mass (*centroid*).



If $y = f(x)$ = "*top*", $y = g(x)$ = "*bottom*", then the center of mass (centroid) is

$$\bar{x} = \frac{1}{Area}\int_a^b x(f(x) - g(x))dx$$

$$\bar{y} = \frac{1}{Area}\int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]dx$$

Example: Find the centroid of the region bounded by $y = x^2$ and $y = 4$.

$$\text{AREA} = \int_{-2}^{2} 4 - x^2\, dx = 4x - \tfrac{1}{3}x^3 \Big|_{-2}^{2}$$

$$= (8 - 8/3) - (-8 + 8/3)$$

$$= 16 - 16/3 = 32/3$$

$$\bar{x} = \frac{1}{32/3}\int_{-2}^{2} x(4 - x^2)\,dx = \boxed{0}$$

$$\bar{y} = \frac{1}{32/3}\int_{-2}^{2} \frac{1}{2}[(4)^2 - (x^2)^2]\,dx$$

$$= \frac{3}{32}\left(\frac{88}{3}\right) = \frac{88}{32} = \frac{11}{4} = \boxed{2.75}$$

Derivativation (don't need to write)

If you are given ***n* points**

(x$_1$,y$_1$), (x$_2$,y$_2$), …, (x$_n$,y$_n$) with masses

m$_1$, m$_2$, …, m$_n$

then

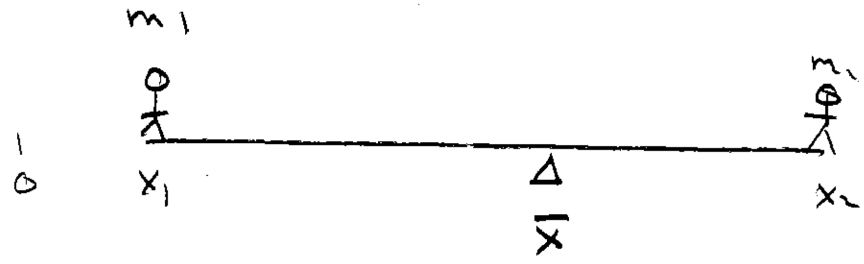
$$M = \text{total mass} = \sum_{i=1}^{n} m_i$$

$$M_y = \text{moment } about\ y \text{ axis} = \sum_{i=1}^{n} m_i x_i$$

$$M_x = \text{moment } about\ x \text{ axis} = \sum_{i=1}^{n} m_i y_i$$

$$\bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1y_1 + \cdots + m_ny_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$



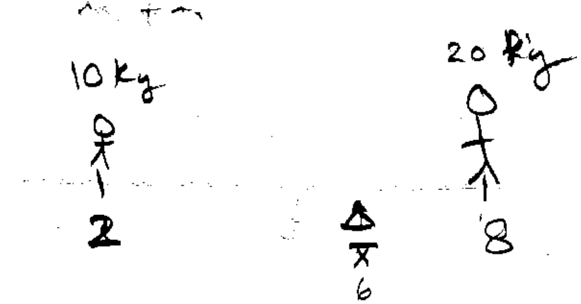
$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} - m_1x_1 = m_2x_2 - m_2\bar{x}$$

$$\Rightarrow (m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2$$

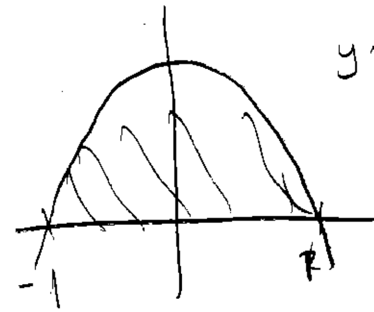
$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Ex)



$$\bar{x} = \frac{(10)\cdot(2) + (20)(8)}{10 + 20} = \frac{180}{30} = \boxed{6}$$

Example:
Find the center of mass (centroid) of a thin plate with uniform density $\rho = 2$ kg/m^2 that looks like the region bounded by $y = 4 - x^2$ and the x-axis.



$y = 1 - x^2$

$$\text{AREA} = \int_{-1}^{1} 1 - x^2 \, dx$$

$$= x - \tfrac{1}{3}x^3 \Big|_{-1}^{1} = (1 - \tfrac{1}{3}) - (-1 + \tfrac{1}{3}) = 2 - \tfrac{2}{3} = \tfrac{4}{3}$$

$$\text{TOTAL MASS} = 2\,\tfrac{kg}{m^2} \cdot \tfrac{4}{3}\,m^2 = \boxed{\tfrac{8}{3}\,kg = M}$$

$$\int_{-1}^{1} x(1 - x^2)\,dx = \int_{-1}^{1} x - x^3 \, dx = 0 \Rightarrow \boxed{M_y = 2(0) = 0}$$

$$\int_{-1}^{1} \tfrac{1}{2}(1 - x^2)^2 \, dx = \tfrac{1}{2}\int_{-1}^{1} 1 - 2x^2 + x^4 \, dx$$

$$= \tfrac{1}{2}\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 \Big|_{-1}^{1}\right]$$

$$= \tfrac{1}{2}\left[(1 - \tfrac{2}{3} + \tfrac{1}{5}) - (-1 + \tfrac{2}{3} - \tfrac{1}{5})\right]$$

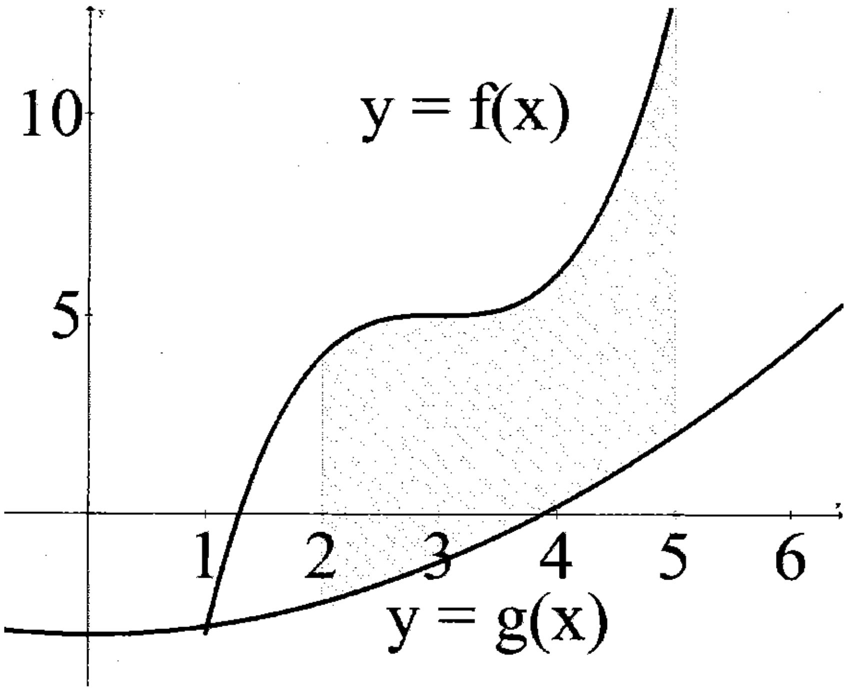
$$= \tfrac{1}{2}\left[2 - \tfrac{4}{3} + \tfrac{2}{5}\right]$$

$$= \tfrac{1}{2}\left[\tfrac{30 - 20 + 6}{15}\right] = \tfrac{1}{2}\,\tfrac{16}{15} = \tfrac{8}{15}$$

$$\boxed{M_x = 2 \cdot \tfrac{8}{15} = \tfrac{16}{15}}$$

$$\boxed{\begin{aligned}\bar{x} &= \frac{M_y}{M} = \frac{0}{8/3} = 0 \\ \bar{y} &= \frac{M_x}{M} = \frac{16/15}{8/3} = \frac{2}{5}\end{aligned}}$$

If the region is bounded between two curves,



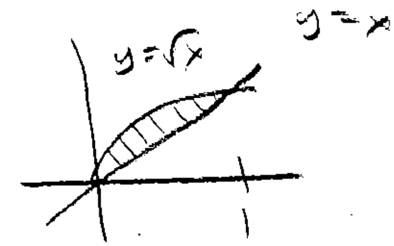
what changes in derivation?

$$\bar{x} = \frac{p\int_a^b x(f(x) - g(x))dx}{p\int_a^b f(x) - g(x)\,dx}$$

$$\bar{y} = \frac{p\int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]dx}{p\int_a^b f(x) - g(x)dx}$$

Example:

Find the center of mass (centroid) of a thin plate with uniform density ρ = 3 kg/m^2 that looks like the region bounded by $y = x$ and $y = \sqrt{x}$.



AREA $= \int_0^1 \sqrt{x} - x \; dx = 1/6 \Rightarrow m = 3/6 = 1/2$

$\int_0^1 x(\sqrt{x} - x)dx = 1/15 \Rightarrow m_y = \frac{3}{15} = 1/5$

$\int_0^1 \frac{1}{2}[(\sqrt{x})^2 - (x)^2]\,dx = \frac{1}{12} \Rightarrow m_x = \frac{3}{12} = 1/4$

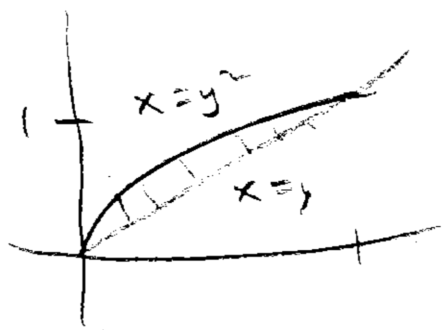
$$\boxed{\begin{aligned}\bar{x} &= \frac{m_y}{m} = \frac{1/5}{1/2} = 2/5 \\ \bar{y} &= \frac{m_x}{m} = \frac{1/4}{1/2} = 1/2\end{aligned}}$$

What if we want to do everything in terms of y instead of x, then swap "x" and "y" ***everywhere*** in these formulas.

f(y) = right bound, g(y) = left bound, then the center of mass is given by

$$\bar{y} = \frac{p\int_a^b y(f(y) - g(y))dy}{p\int_a^b f(y) - g(y)\,dy}$$

$$\bar{x} = \frac{p\int_a^b \frac{1}{2}[(f(y))^2 - (g(y))^2]dy}{p\int_a^b f(y) - g(y)dy}$$



Example:

Find the center of mass (centroid) of a thin plate with uniform density ρ = 3 kg/m^2 that looks like the region bounded by $y = x$ and $y^2 = x$.

(same problem from previous page, but now do it in terms of y.)

Area $= \int_0^1 y - y^2\,dy = \frac{1}{2}y^2 - \frac{1}{3}y^3\Big|_0^1$

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ ✓

$\int_0^1 y(y - y^2)\,dy = \frac{1}{12}$

$\int_0^1 \frac{1}{2}[(y)^2 - (y^2)^2]\,dy = \frac{1}{15}$

$\bar{y} = \frac{1/12}{1/6} = \frac{1}{2}$

$\bar{x} = \frac{1/15}{1/6} = \frac{2}{5}$

Just for your own interest:

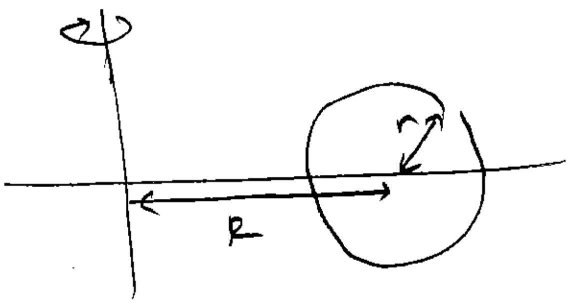
Theorem of Pappus

The volume of a solid of revolution is equal to the product of the area of the region, A, and the distance traveled by the center of mass of the region around the axis of rotation, d. (Note: $d = 2\pi\bar{x}$)

Thus, Volume = (Area)$2\pi\bar{x}$

Quick Application:

Find the volume of the torus.



CENTER OF MASS OF CIRCLE $= (\bar{x}, \bar{y}) = (R, 0)$

Proof

Using the shell method, we get:

$$\text{Volume} = \int_a^b 2\pi x\big(f(x) - g(x)\big)dx$$

$$= 2\pi \int_a^b x\,\big(f(x) - g(x)\big)dx$$

From today:

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x))dx}{\text{Area}}, \qquad \text{so}$$

$$2\pi \int_a^b x(f(x) - g(x))dx = 2\pi\bar{x}(Area).$$

AREA OF CIRCLE $= \pi r^2$

DIST. TRAVELED BY $(R, 0)$ IN ONE REVOLUTION $= 2\pi R$

$(\text{AREA}) \cdot 2\pi\bar{x} = \pi r^2 \cdot 2\pi R = \boxed{2\pi^2 r^2 R}$